Lecture 11 Summary Thurs 14/08/14

Vocabulary

* continuous function (alternative definitions)
* sequence converges pointwise
* sequence converges uniformly

Examples

SEQUENCES OF FUNCTIONS

(1) $f_n : [0,1) \to [0,1)$, $x \mapsto x^n$ for $n \in \mathbb{Z}_{>0}$ $\Big\}$ $\lim_{n\to\infty} f_n = f$ where $f: [0,1) \to [0,1)$, $x \mapsto 0$

(2) $f_n : [0,1] \to [0,1]$, $x \mapsto x^n$ for $n \in \mathbb{Z}_{>0}$ $\Big\}$ $\lim_{n\to\infty} f_n = f$ where $f: [0,1] \to [0,1]$, $x \mapsto \begin{cases} 0, & \text{if } x<1 \\ 1, & \text{if } x=1 \end{cases}$

(3) $f_n : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, $x \mapsto x^n$ for $n \in \mathbb{Z}_{>0}$ $\Big\}$ $\lim_{n\to\infty} f_n$ where $f: \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ <u>does not exist</u>

Homework

- Show that $f: X \to C$ satisfies (1) iff f satisfies (4) where (X,d) and (C,ρ) are viewed as topological spaces with the metric space topology.

  (1) "Let (X,d) and (C,ρ) be metric spaces and $f: X \to C$ a function. The function $f: X \to C$ is continuous iff f satisfies:
  if $x \in X$ and $\varepsilon \in \mathbb{R}_{>0}$ then there exist $\delta \in \mathbb{R}_{>0}$ such that
  if $y \in X$ and $d(x,y) < \delta$ then $\rho(f(x), f(y)) < \varepsilon$."

  (4) "Let $(X,\mathcal{T})$ and $(C,\mathcal{R})$ be topological spaces and $f: X \to C$ a function. The function $f: X \to C$ is continuous iff f satisfies:
  if $\underbrace{V \text{ is open in } C}_{V \in \mathcal{R}}$ then $\underbrace{f^{-1}(V) \text{ is open in } X}_{f^{-1}(V) \in \mathcal{T}}$." $\left(f^{-1}(V) = \{x \in X \mid f(x) \in V\}\right.$ ie. preimage$\left.\right)$

- If $f: X \to C$ is uniformly continuous then f is continuous.

- Given an example of an f which is continuous and <u>not</u> uniformly continuous.

- Show that $\mathbb{R} \to \mathbb{R}_{\geq 0}$, $x \mapsto \frac{1}{1+x^2}$ is uniformly continuous.

- Show that $\mathbb{R} \to \mathbb{R}$, $x \mapsto x^2$ is continuous but <u>not</u> uniformly continuous.